Chapter 15: Equations

Equation 15.1:

$$\mathrm{P}\left[\left(y_i < 50,000 | 12 \text{ years of schooling}\right) \cap \left(y_j \geq 50,000 | 16 \text{ years of schooling}\right)\right]$$

Equation 15.2:

$$\begin{aligned} &\mathrm{P}\left[\left(y_i < 50,000 | 12 \text{ years of schooling}\right) \cap \left(y_j \geq 50,000 | 16 \text{ years of schooling}\right)\right] \\ &= \mathrm{P}\left(y_i < 50,000 | 12 \text{ years of schooling}\right) \times \mathrm{P}\left(y_j \geq 50,000 | 16 \text{ years of schooling}\right) \end{aligned}$$

Equation 15.3:

$$\mathrm{P}\begin{bmatrix} \left(y_1 < 50,000 | x_1\right) \cap \left(y_2 < 50,000 | x_2\right) \cap \cdots \cap \left(y_k < 50,000 | x_k\right) \\ \cap \left(y_{k+1} \geq 50,000 | x_{k+1}\right) \cap \left(y_{k+2} \geq 50,000 | x_{k+2}\right) \cap \cdots \cap \left(y_n \geq 50,000 | x_n\right) \end{bmatrix}$$

Equation 15.4:

$$\begin{aligned} &\mathrm{P}\begin{bmatrix} \left(y_1 < 50,000 | x_1\right) \cap \left(y_2 < 50,000 | x_2\right) \cap \cdots \cap \left(y_k < 50,000 | x_k\right) \\ \cap \left(y_{k+1} \geq 50,000 | x_{k+1}\right) \cap \left(y_{k+2} \geq 50,000 | x_{k+2}\right) \cap \cdots \cap \left(y_n < 50,000 | x_n\right) \end{bmatrix} \\ &= \mathrm{P}\left(y_1 < 50,000 | x_1\right) \times \mathrm{P}\left(y_2 < 50,000 | x_2\right) \times \cdots \times \mathrm{P}\left(y_k < 50,000 | x_k\right) \\ &\quad \times \mathrm{P}\left(y_{k+1} \geq 50,000 | x_{k+1}\right) \times \mathrm{P}\left(y_{k+2} \geq 50,000 | x_{k+2}\right) \times \cdots \times \mathrm{P}\left(y_n < 50,000 | x_n\right) \\ &= \left[\prod_{i=1}^{k} \mathrm{P}\left(y_i < 50,000 | x_i\right)\right] \times \left[\prod_{i=k+1}^{n} \mathrm{P}\left(y_i \geq 50,000 | x_i\right)\right] \end{aligned}$$

Equation 15.5:

$$y_i^* = \alpha^* + \beta x_i + \varepsilon_i$$

Equation 15.6:

$$y_i = 1 \text{ if } y_i^* = \alpha^* + \beta x_i + \varepsilon_i \geq T$$

Equation 15.7:

$$y_i = 0 \text{ if } y_i^* = \alpha^* + \beta x_i + \varepsilon_i < T$$

Equation 15.8:

$$y_i = 1 \text{ if } y_i^* = \left(\alpha^* - T\right) + \beta x_i + \varepsilon_i \geq 0$$

Equation 15.9:

$$y_i = 0 \text{ if } y_i^* = \left(\alpha^* - T\right) + \beta x_i + \varepsilon_i < 0$$

Equation 15.10:

$$y_i = 1 \text{ if } y_i^* = \alpha + \beta x_i + \varepsilon_i \geq 0$$

Equation 15.11:

$$y_i = 0 \text{ if } y_i^* = \alpha + \beta x_i + \varepsilon_i < 0$$

Equation 15.12:

$$y_i = 1 \text{ if } \varepsilon_i \geq -\left(\alpha + \beta x_i\right)$$

Equation 15.13:

$$y_i = 0 \text{ if } \varepsilon_i < -\left(\alpha + \beta x_i\right)$$

Equation 15.14:

$$\mathrm{P}\left(y_i = 1 \mid x_i\right) = \mathrm{P}\left(\varepsilon_i \geq -\left(\alpha + \beta x_i\right)\right)$$

Equation 15.15:

$$\mathrm{P}\left(y_i = 0 \mid x_i\right) = \mathrm{P}\left(\varepsilon_i < -\left(\alpha + \beta x_i\right)\right)$$

Equation 15.16:

$$\mathrm{P}\left(y_i = 1 \mid x_i\right) = \mathrm{P}\left(\frac{\varepsilon_i - 0}{\sigma} \geq \frac{-\left(\alpha + \beta x_i\right) - 0}{\sigma}\right) = \mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq -\left(\frac{\alpha}{\sigma} + \frac{\beta}{\sigma} x_i\right)\right)$$

Equation 15.17:

$$\mathrm{P}\left(y_i = 0 \mid x_i\right) = \mathrm{P}\left(\frac{\varepsilon_i - 0}{\sigma} < \frac{-\left(\alpha + \beta x_i\right) - 0}{\sigma}\right) = \mathrm{P}\left(\frac{\varepsilon_i}{\sigma} < -\left(\frac{\alpha}{\sigma} + \frac{\beta}{\sigma} x_i\right)\right)$$

Equation 15.18:

$$\mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq -\left(\frac{\alpha}{\sigma} + \frac{\beta}{\sigma} x_i\right)\right)$$

Equation 15.19:

$$\mathrm{P}\left(\frac{\varepsilon_i}{\sigma} < -\left(\frac{\alpha}{\sigma} + \frac{\beta}{\sigma} x_i\right)\right)$$

Equation 15.20:

$$\mathrm{P}\left(\text{observing the sample}\right) = \left[\prod_{y_i=0} \mathrm{P}\left(\frac{\varepsilon_i}{\sigma} < -\left(\frac{\alpha}{\sigma} + \frac{\beta}{\sigma} x_i\right)\right)\right]\left[\prod_{y_i=1} \mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq -\left(\frac{\alpha}{\sigma} + \frac{\beta}{\sigma} x_i\right)\right)\right]$$

Equation 15.21:

$$\mathrm{L}\left(y_1, y_2, \ldots, y_n, a, b\right) = \left[\prod_{y_i=0} \mathrm{P}\left(\frac{\varepsilon_i}{\sigma} < -\left(a + bx_i\right)\right)\right]\left[\prod_{y_i=1} \mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq -\left(a + bx_i\right)\right)\right]$$

Equation 15.22:

$$\mathrm{P}\left(\frac{\varepsilon_i}{\sigma} < -\left(a + bx_i\right)\right) \text{ or } \mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq -\left(a + bx_i\right)\right)$$

Equation 15.23:

$$\ln \mathrm{L}\left(y_1, y_2, \ldots, y_n, a, b\right) = \left[\sum_{y_i=0} \ln \mathrm{P}\left(\frac{\varepsilon_i}{\sigma} < -\left(a + bx_i\right)\right)\right]\left[\sum_{y_i=1} \ln \mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq -\left(a + bx_i\right)\right)\right]$$

Equation 15.24:

$$\frac{\partial\left(\ln \mathrm{L}\left(y_1, y_2, \ldots, y_n, a, b\right)\right)}{\partial a} = \left[\sum_{y_i=0} \frac{\partial\left(\ln \mathrm{P}\left(\frac{\varepsilon_i}{\sigma} < -\left(a+bx_i\right)\right)\right)}{\partial a}\right]\left[\sum_{y_i=1} \frac{\partial\left(\ln \mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq -\left(a+bx_i\right)\right)\right)}{\partial a}\right]$$

Equation 15.25:

$$\frac{\partial\left(\ln \mathrm{L}\left(y_1, y_2, \ldots, y_n, a, b\right)\right)}{\partial b} = \left[\sum_{y_i=0} \frac{\partial\left(\ln \mathrm{P}\left(\frac{\varepsilon_i}{\sigma} < -\left(a+bx_i\right)\right)\right)}{\partial b}\right]\left[\sum_{y_i=1} \frac{\partial\left(\ln \mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq -\left(a+bx_i\right)\right)\right)}{\partial b}\right]$$

Equation 15.26:

$$\mathrm{P}\left(\frac{\varepsilon_i}{\sigma} < -\left(a+bx_i\right)\right) = \Phi\left(-\left(a+bx_i\right)\right)$$

Equation 15.27:

$$\mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq -\left(a+bx_i\right)\right) = 1 - \Phi\left(-\left(a+bx_i\right)\right)$$

Equation 15.28:

$$\frac{\partial\left(\ln \mathrm{P}\left(\frac{\varepsilon_i}{\sigma} < -\left(a+bx_i\right)\right)\right)}{\partial a} = \frac{\varphi\left(-\left(a+bx_i\right)\right)}{\Phi\left(-\left(a+bx_i\right)\right)}$$

Equation 15.29:

$$\frac{\partial\left(\ln \mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq -\left(a+bx_i\right)\right)\right)}{\partial a} = \frac{-\varphi\left(-\left(a+bx_i\right)\right)}{1-\Phi\left(-\left(a+bx_i\right)\right)}$$

Equation 15.30:

$$\frac{\partial\left(\ln \mathrm{L}\left(y_1, y_2, \ldots, y_n, a, b\right)\right)}{\partial a} = \left[\sum_{y_i=0} \frac{\varphi\left(-\left(a+bx_i\right)\right)}{\Phi\left(-\left(a+bx_i\right)\right)}\right]\left[\sum_{y_i=1} \frac{-\varphi\left(-\left(a+bx_i\right)\right)}{1-\Phi\left(-\left(a+bx_i\right)\right)}\right]$$

Equation 15.31:

$$\left[\sum_{y_i=0} \frac{\varphi\left(-\left(a+bx_i\right)\right)}{\Phi\left(-\left(a+bx_i\right)\right)}\right]\left[\sum_{y_i=1} \frac{-\varphi\left(-\left(a+bx_i\right)\right)}{1-\Phi\left(-\left(a+bx_i\right)\right)}\right] = 0$$

Equation 15.32:

$$\frac{\partial\left(\ln \mathrm{P}\left(\frac{\varepsilon_i}{\sigma} < -\left(a+bx_i\right)\right)\right)}{\partial b} = \frac{\varphi\left(-\left(a+bx_i\right)\right)}{\Phi\left(-\left(a+bx_i\right)\right)} x_i$$

Equation 15.33:

$$\frac{\partial\left(\ln \mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq -\left(a+bx_i\right)\right)\right)}{\partial b} = \frac{-\varphi\left(-\left(a+bx_i\right)\right)}{1-\Phi\left(-\left(a+bx_i\right)\right)} x_i$$

Equation 15.34:

$$\frac{\partial\left(\ln \mathrm{L}\left(y_1, y_2, \ldots, y_n, a, b\right)\right)}{\partial b}=\left[\sum_{y_i=0} \frac{\varphi\left(-\left(a+b x_i\right)\right)}{\Phi\left(-\left(a+b x_i\right)\right)} x_i\right]\left[\sum_{y_i=1} \frac{-\varphi\left(-\left(a+b x_i\right)\right)}{1-\Phi\left(-\left(a+b x_i\right)\right)} x_i\right]$$

Equation 15.35:

$$\left[\sum_{y_i=0} \frac{\varphi\left(-\left(a+b x_i\right)\right)}{\Phi\left(-\left(a+b x_i\right)\right)} x_i\right]\left[\sum_{y_i=1} \frac{-\varphi\left(-\left(a+b x_i\right)\right)}{1-\Phi\left(-\left(a+b x_i\right)\right)} x_i\right]=0$$

Equation 15.36:

$$\mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq-\left(a+b x_i\right)\right)$$

Equation 15.37:

$$\mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq-\left(a+b\left(\bar{x}+\Delta x\right)\right)\right)-\mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq-\left(a+b \bar{x}\right)\right)$$

Equation 15.38:

$$\frac{\partial\left(\mathrm{P}\left(y_i=1 \mid x_i\right)\right)}{\partial x_i}=\frac{\partial\left(\mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq-\left(a+b\right)\right)\right)}{\partial x_i}=\varphi\left(-\left(a+b x_i\right)\right) b$$

Equation 15.39:

$$\mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq-\left(-3.555+.1854(0)\right)\right)=\mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq-\left(-3.555\right)\right)=\mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq 3.555\right)$$

Equation 15.40:

$$\mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq -\left(-3.355 + .1854(12)\right)\right) = \mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq -\left(-3.355 + 2.225\right)\right) = \mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq 1.130\right) = .1292$$

Equation 15.41:

$$\mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq -\left(-3.355 + .1854(13)\right)\right) = \mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq -\left(-3.355 + 2.410\right)\right) = \mathrm{P}\left(\frac{\varepsilon_i}{\sigma} \geq .945\right) = .1724$$

Equation 15.42:

$$\mathrm{P}\left(y_i = 1 \mid x_i\right) = \mathrm{P}\left(\varepsilon_i \geq -\left(\alpha + \sum_{j=1}^{k} \beta_j x_{ji}\right)\right)$$

Equation 15.43:

$$w_i = \delta + \sum_{l=1}^{m} \gamma_l z_{li} + \nu_i$$

Equation 15.44:

$$\frac{\partial\left(\ln \mathrm{P}\left(\frac{\varepsilon}{\sigma} < -\left(a + bx_i\right)\right)\right)}{\partial a} = \frac{e^{-\left(a+bx_i\right)}}{\left(1 + e^{-\left(a+bx_i\right)}\right)^2}$$

Equation 15.45:

$$\frac{\partial\left(\ln \mathrm{P}\left(\frac{\varepsilon}{\sigma} \geq -\left(a + bx_i\right)\right)\right)}{\partial a} = 1 - \frac{e^{-\left(a+bx_i\right)}}{\left(1 + e^{-\left(a+bx_i\right)}\right)^2}$$

Equation 15.46:

$$\frac{\partial\left(\ln \mathrm{P}\left(\frac{\varepsilon_i}{\sigma} < -\left(a+bx_i\right)\right)\right)}{\partial b} = \frac{e^{-\left(a+bx_i\right)}}{\left(1+e^{-\left(a+bx_i\right)}\right)^2}x_i$$

Equation 15.47:

$$\frac{\partial\left(\ln \mathrm{P}\left(\frac{\varepsilon}{\sigma} \geq -\left(a+bx_i\right)\right)\right)}{\partial b} = 1-\frac{e^{-\left(a+bx_i\right)}}{\left(1+e^{-\left(a+bx_i\right)}\right)^2}x_i$$

Equation 15.48:

$$\mathrm{V}\left(\varepsilon_i\right) = \mathrm{P}\left(y_i = 1 \mid x_i\right)\mathrm{P}\left(y_i = 0 \mid x_i\right)$$

Equation 15.49:

$$\mathrm{L}\left(y_1, y_2, \ldots, y_n\right) = g(y_1)g(y_2), \ldots, g(y_n) = \prod_{i=1}^{n} g(y_i)$$

Equation 15.50:

$$\mathrm{L}\left(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\right) = f(\varepsilon_1)f(\varepsilon_2), \ldots, f(\varepsilon_n) = \prod_{i=1}^{n} f(\varepsilon_i)$$

Equation 15.51:

$$\ln \mathrm{L} = \ln\left[\mathrm{L}\left(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\right)\right] = \sum_{i=1}^{n} \ln f(\varepsilon_i)$$

Equation 15.52:

$$\varphi(\varepsilon_i) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2} \frac{\left(y_i - \alpha - \beta x_i\right)^2}{\sigma^2} \right)$$

Equation 15.53:

$$\ln \mathrm{L}\left(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\right) = \sum_{i=1}^{n} \left(-\ln\sigma - \frac{1}{2}\ln\left(2\pi\right) - \frac{1}{2} \frac{\left(y_i - \alpha - \beta x_i\right)^2}{\sigma^2} \right)$$

Equation 15.54:

$$\ln \mathrm{L}\left(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\right) = \sum_{i=1}^{n} \left(-\ln\sigma\right) + \sum_{i=1}^{n} \left(-\frac{1}{2}\ln\left(2\pi\right) \right) + \sum_{i=1}^{n} \left(-\frac{1}{2} \frac{\left(y_i - \alpha - \beta x_i\right)^2}{\sigma^2} \right)$$

Equation 15.55:

$$\ln \mathrm{L}\left(\varepsilon_i, \varepsilon_2, \ldots, \varepsilon_n : a_{\mathrm{ML}}, b_{\mathrm{ML}}, s_{\mathrm{ML}}\right) =$$
$$- n \ln s_{\mathrm{ML}} - \frac{n}{2}\ln\left(2\pi\right) - \frac{1}{2s_{\mathrm{ML}}^2} \sum_{i=1}^{n} \left(y_i - a_{\mathrm{ML}} - b_{\mathrm{ML}} x_i\right)^2$$